

Improved approximative multicoloring of hexagonal graphs

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Abstract

In 1999, McDiarmid and Reed conjectured that the approximation ratio $9/8$ of multichromatic number to weighted clique number asymptotically is the best possible for general weighted hexagonal graphs. We prove that there is a proper multicoloring of G that uses at most $15\lfloor \frac{\omega(G)}{12} \rfloor + 18$ colors improving the best previously known asymptotic ratio from $4/3$ to $5/4$.

1 Introduction

A fundamental problem concerning cellular networks is to assign sets of frequencies (colors) to transmitters (vertices) in order to avoid unacceptable interferences [1]. The number of frequencies' demands from a transmitter may vary between transmitters. In a usual cellular model, transmitters are centers of hexagonal cells and the corresponding adjacency graph is a subgraph of the infinite triangular lattice. An integer $d(v)$ is assigned to each vertex of the triangular lattice and will be called the *weight* (or *demand*) of the vertex v . The vertex weighted graph induced on the subset of vertices of the triangular lattice is called a *hexagonal graph*, and is denoted by $G = (V, E, d)$. A *proper multicoloring* of G is a mapping f from $V(G)$ to subsets of integers such that $|f(v)| \geq d(v)$ for any vertex $v \in V(G)$ and $f(v) \cap f(u) = \emptyset$ for any pair of adjacent vertices u and v in the graph G . The minimal cardinality of a proper multicoloring of G is called the *multichromatic number*, denoted $\chi(G)$ (for simplicity, as it is the only chromatic number studied here). Another invariant of interest in this context is the (*weighted*) *clique number*, $\omega(G)$, defined as follows: The weight of a clique of G is the sum of weights on its vertices and $\omega(G)$ is the

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maximal clique weight on G . Clearly, $\chi(G) \geq \omega(G)$ and a simple example shows that the upper bound is $\chi(G) \leq (4/3)\omega(G)$ (see Figure 1). However, note that in the example, $\omega(G) = 3$ and $\chi(G) = 4$ and we are not aware of any example with $\omega(G) > 3$ and $\chi(G) = 4\omega(G)/3$. Interesting enough, the best known approximation algorithms for hexagonal graphs give solutions within $\chi(G) \leq (4/3)\omega(G) + O(1)$ [6, 7, 12]. For a later reference note that the constant is very small, more precisely, we have $\chi(G) \leq \frac{4\omega(G)+1}{3} \leq 4\lceil \frac{\omega(G)}{3} \rceil$.

Better bounds have been obtained for triangle-free hexagonal graphs. In [3] a distributed algorithm with competitive ratio $5/4$ is given. Existence of 2-local distributed algorithm with competitive ratio $5/4$ is provided in [11], and an inductive proof for ratio $7/6$ is reported in [2]. A 2-local $7/6$ -competitive algorithm for a subclass of triangle-free hexagonal graphs is given in [13]. As the shortest odd cycle that can be realized as an induced subgraph of triangular lattice has length 9, the best possible approximation is $\chi(G) \leq (9/8)\omega(G) + O(1)$. A challenging research question is to settle the conjecture due to McDiarmid and Reed:

Conjecture [6] There is an absolute constant c such that for every weighted hexagonal graph G ,

$$\chi(G) \leq \frac{9}{8}\omega(G) + c$$

The conjecture remains unsolved even in the relaxed version for triangle-free hexagonal graphs.

An interesting special case of a proper multicoloring is when d is a constant function. In this case we define a n - $[k]$ -coloring to be an assignment of sets of k colors from a set of n colors to each vertex. For example a 9- $[4]$ -coloring is an assignment of four colors from the set $\{1, \dots, 9\}$ to each vertex. In [3], a 5- $[2]$ -coloring algorithm is used to provide a general $5/4$ -competitive algorithm. An elegant idea that implies the existence of a 14- $[6]$ -coloring was presented in [14]. The existence of a 7- $[3]$ -coloring follows from result of [2]. A shorter proof based on the 4-colour theorem is provided in [8]. In [9] a linear time algorithm for 7- $[3]$ -coloring of an arbitrary triangle-free hexagonal graph G is given. The only result regarding 9- $[4]$ -coloring, however restricted to a subclass of hexagonal graphs, is the algorithm for 9- $[4]$ -coloring of triangle-free hexagonal graphs without neighboring corners [16]. The n - $[k]$ -colorings can be used to obtain $\frac{n}{2k}$ approximations for $\chi(G)$ due to the following lemma. (Folklore, for example the proof for $k = 4$ appears in [16].)

Lemma 1.1 *For each triangle-free graph G the following statements are equivalent:*

- (i) G is $(2k + 1)$ - $[k]$ -colorable,
- (ii) G is $\lceil \frac{2k+1}{2k}\omega(G) \rceil$ multicolorable.

On the other hand, triangle-free hexagonal graphs with large odd girth surely allow better approximation, since hexagonal graphs are planar. However, we are

not aware of any work on hexagonal graphs that would improve the well known results that hold for arbitrary planar graphs [4, 5]. Another avenue that may be of interest is to study generalization of the (planar) hexagonal graphs to 3D, where the triangular grid is replaced by the "cannonball" grid [10].

Here we consider the general hexagonal graphs, and concentrate on improvement of the approximation bound. The basic idea is to partition the original hexagonal graph into three triangle-free hexagonal graphs and apply the algorithms for triangle-free graphs. Roughly speaking, assuming that the weights can be evenly divided among the three subgraphs, then any approximation bound for the triangle-free hexagonal graphs implies the same approximation bound for the general case. The main new result presented here is Theorem 3.2 implying that there is an algorithm that colors any weighted hexagonal graph with $\chi(G) \leq \frac{5}{4}\omega(G) + O(1)$ colors. This significantly improves the previously best known bound $\chi(G) \leq \frac{4}{3}\omega(G) + O(1)$ [6, 7, 12]. The main idea might be used to improve the bound to get closer to the conjectured $\chi(G) \leq \frac{9}{8}\omega(G) + O(1)$.

The rest of the paper is organized as follows. In the next section, we formally define some basic terminology. The general method and its application using a $5/4$ approximation algorithm providing the new bound is given in Section 3. The last section provides a short conclusion and discusses ideas for future work.

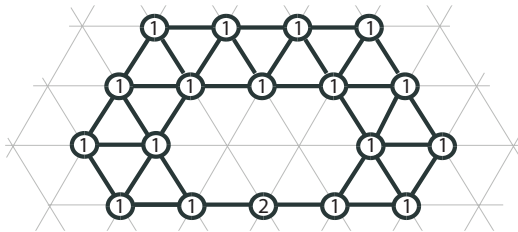


Figure 1: An example of a hexagonal graph.

2 Basic definitions and useful facts

A vertex weighted graph is given by a triple $G(E, V, d)$, where V is the set of vertices, E is the set of edges and $d : V \rightarrow \mathbb{N}$ is a weight function assigning integer weights to vertices of G .

The vertices of the triangular lattice T can be described as follows: the position of each vertex is an integer linear combination $x\vec{p} + y\vec{q}$ of two vectors $\vec{p} = (1, 0)$ and $\vec{q} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Thus vertices of the triangular lattice may be identified with pairs (x, y) of integers. Given the vertex v we will refer to its coordinates as $x(v)$ and $y(v)$. Two vertices are adjacent when the Euclidean distance between them is one. Therefore each vertex (x, y) has six neighbors: $(x - 1, y)$, $(x - 1, y + 1)$, $(x, y + 1)$, $(x + 1, y)$, $(x + 1, y - 1)$, $(x, y - 1)$. For convenience we refer to the neighbors as:

left, up-left, up-right, right, down-right and down-left neighbor. Assume that we are given a weight function $d : V \rightarrow \{0, 1, 2, \dots\}$ on vertices of triangular lattice. We define a *weighted hexagonal graph* $G = (V, E, d)$ as an induced subgraph on vertices of positive demand on the triangular lattice (see Figure 2). Sometimes we want to work with (unweighted) hexagonal graphs $G = (V, E)$ that can be defined as induced graphs on the subset of vertices of the triangular lattice. In such a graph we can say that the weight function in each vertex in the graph is equal to 1, and for vertices out of the graph it is equal to 0.

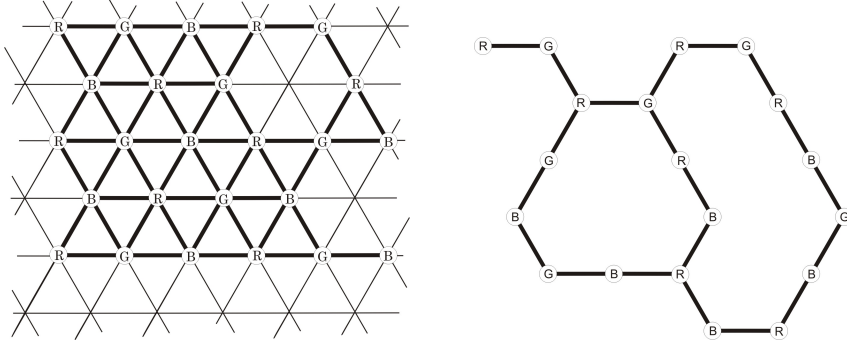


Figure 2: An example of a hexagonal graph and a triangle-free hexagonal graph (with base colorings).

There exists an obvious 3-coloring of the infinite triangular lattice which gives the partition of the vertex set of any hexagonal graph into three independent sets. Let us denote the color of vertex v in this 3-coloring by $bc(v)$ and call it a *base color* (for simplicity we will use *red*, *green* and *blue* as base colors and their arrangement is given in Figure 2), i.e. $bc(v) \in \{\text{red}, \text{blue}, \text{green}\}$.

We call a *triangle-free hexagonal graph* an induced subgraph of the triangular lattice without a 3-clique (see Figure 2).

3 The general method and the 5/4 approximation

The main idea is to partition the weighted hexagonal graph into three triangle-free hexagonal (sub)graphs such that the weights are divided among the partition so that the clique number of each subgraph is about one third of the original clique number. More formally, given a weighted hexagonal graph $G(V, E, d)$, the idea is to construct triangle-free subgraphs $G_1(V_1, E_1, d_1)$, $G_2(V_2, E_2, d_2)$, and $G_3(V_3, E_3, d_3)$, such that G_i is induced on V_i (for $i \in \{1, 2, 3\}$), and $d_1(v) + d_2(v) + d_3(v) = d(v)$ for any $v \in V$. Among the decompositions we prefer the decompositions for which $\omega(G_1) + \omega(G_2) + \omega(G_3) = \omega(G)$, or, at least $\omega(G_1) + \omega(G_2) + \omega(G_3) = \omega(G) + C$ for some absolute constant C . Having this, the second step would be to multicolor with an approximation algorithm each of the subgraphs that were constructed, therefore

obviously we would obtain the same approximation ratio.

At present, we are not able to provide such a decomposition in general case. However, below we show how to decompose a special class of weighted hexagonal graphs which in turn enables an iterative method that achieves $5/4$ approximation for general hexagonal graphs.

In the rest of this section, we will first provide a decomposition of a special class of weighted hexagonal graph into three triangle-free weighted hexagonal graphs. For these graphs, clique number is 12 by definition, and we show how to multicolor them with at most 15 colors. Finally, by iterated application of the decomposition that reduces the clique number we prove the main result, Theorem 3.2.

3.1 A special class of weighted hexagonal graphs

For a weighted hexagonal graph $\tilde{G} = (\tilde{V}, \tilde{E}, \tilde{d})$ considered in this section we assume that the vertices of \tilde{G} have weight $\tilde{d}(v) = 4$ when v is on a triangle. A vertex that is not on any triangle has weight $\tilde{d}(v) = 4, 6$, or 8 so that the weight on any clique is at most 12. (Hence, any neighbor of a vertex of weight 8 must have weight 4, while a neighbor of a vertex of weight 6 may have weight 6 or 4.) Finally, if v is an isolated vertex, then $\tilde{d}(v) = 12$. Say a vertex v is *light* if $\tilde{d}(v) < 6$.

Construct the subgraphs G_{RB} , G_{BG} , and G_{GR} of \tilde{G} as follows. Recall the basic 3-coloring. The subgraph G_{RB} is induced on the vertex set V_{RB} given by the following two rules: (1) If a vertex $v \in V(\tilde{G})$ lies in a triangle, and is of base color R or B , then $v \in V_{RB}$. (2) If a vertex v has weight at least 6, then $v \in V_{RB}$. Or, in other words: V_{RB} is $V(\tilde{G})$ without light vertices of base color green G . Clearly, G_{RB} is triangle-free.

Triangle-free subgraphs G_{BG} and G_{GR} are defined analogously. (They are subgraphs of \tilde{G} induced on the vertex set of \tilde{G} without light vertices of the corresponding base color.)

Observation 3.1 *Assume the subgraphs G_{RB} , G_{BG} , and G_{GR} of \tilde{G} are constructed as explained above. Then the following is true: (1) Each vertex of G that lies on a triangle appears in exactly two subgraphs. (2) Each vertex that is not light and hence does not lie on a triangle is in all 3 subgraphs.*

Define the weights for each $G_\star \in \{G_{RB}, G_{BG}, G_{GR}\}$ as follows:

- Any isolated vertex in G_\star gets weight 4.
- A vertex of G_\star that lies on a triangle in \tilde{G} gets weight 2.
- A vertex v that is not on a triangle in \tilde{G} is assigned weight
 - $\tilde{d}(v) = 2$ if it has more than one neighbor in G_\star ,

- $\tilde{d}(v) = 3$ if it has exactly one neighbor in G_\star , and
- $\tilde{d}(v) = 4$ if it is isolated in G_\star .

It is easy to see that for any vertex $v \in \tilde{V}$, the sum of weights assigned to v in G_{RB} , G_{BG} , and G_{GR} is at least $\tilde{d}(v)$.

Some cases are trivial, namely: If v is on a triangle in \tilde{G} , then it is assigned demand 2 in each of the subgraphs, and altogether $2+2+2 = 6 = \tilde{d}(v)$ as needed. If v is isolated in \tilde{G} , then it is isolated in each of the subgraphs, and hence $4+4+4 = 12 = \tilde{d}(v)$.

If v is not on a triangle and is not isolated in \tilde{G} , then it may have weight 4, 6, or 8. A light vertex ($\tilde{d}(v) = 4$) appears in two subgraphs, and it gets $2+2=4$ colors. A vertex of weight 6 appears in all three subgraphs, and therefore receives $2+2+2=6$, which is sufficient. The case when $\tilde{d}(v) = 8$ is settled by the following lemma showing that in this case the demand 8 can always be partitioned, either as $4+2+2$ or as $3+3+2$.

Lemma 3.1 *If a vertex of \tilde{G} has weight 8, then it is either isolated in one of the subgraphs, or it is a vertex of degree one in at least two of the subgraphs.*

Proof: Recall that weight 8 is assigned to vertices that are not on any triangle, and have only neighbors of weight 4. Consider a vertex v with weight 8, and wlog assume it is of base color red. As v is not on a triangle, it may have degree at most 3.

First, assume the vertex of weight 8 has exactly two neighbors of different base colors (see Figure 3 (a)). By definition, the green neighbor is removed in the graph G_{RB} and hence v has only one neighbor in the graph G_{RB} . Similarly, the blue neighbor is removed in the graph G_{GR} and hence v has only one neighbor in the graph G_{GR} . Therefore v is a vertex of degree one in two of the subgraphs as claimed.

The second case is when all neighbors of v are of the same base color. There can be one (Figure 3 (b)), two, or three neighbors (Figure 3 (c)). Wlog say the neighbors are of base color blue. Then v is isolated in the graph G_{GR} , because the blue neighbors are removed in the graph G_{GR} . \square

Lemma 3.2 *There is a proper multicoloring of \tilde{G} with 15 colors.*

Proof: First, use an algorithm for 5-[2] multicoloring [3] to assign two colors to each vertex of demand 2 of the graphs G_{RB} , G_{BG} , and G_{GR} . (Use three distinct sets of five colors.) This assigns four colors to each vertex of \tilde{G} that lies on a triangle and up to six colors to vertices that appear in all three subgraphs (in particular, it provides exactly six colors to vertices that have demand 2 in all three subgraphs).

Next, observe that vertices of demand 3 in G_\star have at most one neighbor, so there are 3 free colors to assign.

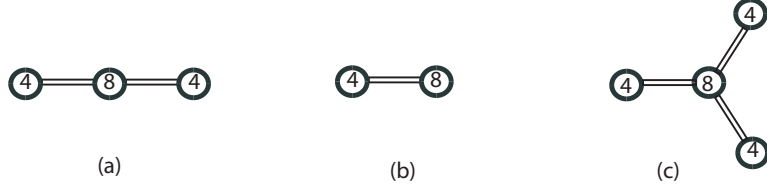


Figure 3: A configuration in which the heavy vertex ($w = 8$) has two neighbors of different base colors (a) and two configurations in which the heavy vertex has neighbors of the same base color (b) and (c).

Finally, note that the vertices of demand 4 are isolated in G_* , hence they can trivially be assigned any set of 4 colors (out of available 5 colors).

Hence, each of the graphs G_{RB} , G_{BG} , and G_{GR} is properly colored using 5 colors, and consequently \tilde{G} we have a proper coloring of \tilde{G} with 15 colors. \square

Example. Consider the graph on Figure 4(a), and three subgraphs on 4(b), (c) and (d). If, for example, the base color $bc(v)$ is red, then the subgraphs (b), (c) and (d) are G_{GR} , G_{BG} , and G_{RB} , respectively. The heavy vertex v has degree one in two of the subgraphs (b) and (d), and hence it can be assigned three colors in each of these two subgraphs. Thus we have a proper multicoloring using 15 colors.

3.2 5/4 approximation algorithm

Theorem 3.1 *Let G be a weighted hexagonal graph and assume all weights are even numbers. There is a proper multicoloring of G that uses at most $15\lfloor \frac{\omega(G)}{12} \rfloor + 15$ colors.*

Proof: If $\omega(G) < 12$ then we can apply any 4/3-approximation algorithm to obtain a proper coloring of G with at most 15 colors. ($\omega(G) \leq 11$ gives $\frac{4\omega(G)+1}{3} \leq 15$.)

Let G be a weighted hexagonal graph with $\omega(G) \geq 12$. Define a sequence of subgraphs of G_i as follows. Start with $G_0 = G$.

In G_i , let $L_i \subseteq V_i$ be the set of vertices with low weight, $L_i = \{v \in V_i \mid d_i(v) < 4\}$. Define the subgraph \tilde{G}_i to be induced graph on \tilde{V}_i , where $\tilde{V}_i = V_i - L_i$ is the subset of V_i of vertices with $d_i(v) \geq 4$. For a vertex v which is isolated in \tilde{G}_i set $\tilde{d}_i(v) = 12$ and let $\tilde{d}_i(v)$ equal 4 on vertices that lie on a triangle. For vertices that are not isolated and not on a triangle, let

$$\begin{aligned} \tilde{d}_i(v) &= 4 \text{ if } d_i(v) \leq \frac{\omega(G_i)}{3}, \\ \tilde{d}_i(v) &= 6 \text{ if } \frac{\omega(G_i)}{3} < d_i(v) < \frac{2\omega(G_i)}{3}, \\ \tilde{d}_i(v) &= 8 \text{ if } d_i(v) \geq \frac{2\omega(G_i)}{3}. \end{aligned}$$

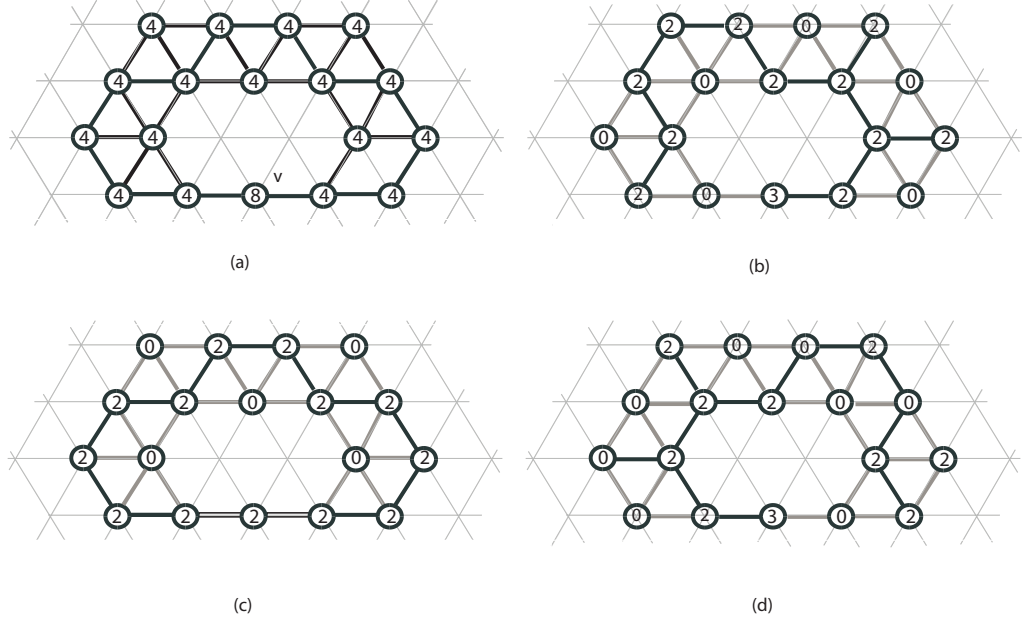


Figure 4: The example: a hexagonal graph (a) and its three triangle-free subgraphs (b), (c), and (d).

Observe that \tilde{G}_i can be colored with 15 colors. This follows from Lemma 3.2 because, by construction, all weights of vertices in \tilde{G}_i are either 4, 6, 8, or 12, and $\omega(\tilde{G}_i) = 12$.

Define G_{i+1} as follows. For $v \in \tilde{V}_i$ let $d_{i+1}(v) = d_i(v) - \tilde{d}_i(v)$ (reduce the weights of vertices in $L_i \subseteq V_i$ and leave the demands of other vertices unchanged).

Claim. $\omega(G_{i+1}) \leq \omega(G_i) - 12$ provided $\omega(G_i) \geq 12$.

To see correctness of the claim, assume $\omega(G_i) \geq 12$. The isolated vertices and triangles of G_i obviously get 12 colors, hence the weight of these cliques is reduced by 12. Now consider a maximal clique in G_i that has exactly two vertices u and v , i.e. $d_i(u) + d_i(v) = \omega(G_i)$. Distinguish three cases. (1) $\tilde{d}_i(u) = 4$. Then $d_i(u) \leq \frac{\omega(G_i)}{3}$, and therefore $d_i(v) \geq \frac{2\omega(G_i)}{3}$, and, by definition, $\tilde{d}_i(v) = 8$. (2) $\tilde{d}_i(u) = 6$. From $\frac{\omega(G_i)}{3} < d_i(v) < \frac{2\omega(G_i)}{3}$ and $d_i(u) + d_i(v) = \omega(G_i)$ we get $\frac{\omega(G_i)}{3} < d_i(u)$ and $d_i(u) < \frac{2\omega(G_i)}{3}$, hence $\tilde{d}_i(v) = 6$. (3) Finally, $\tilde{d}_i(u) = 8$ implies $\tilde{d}_i(v) = 4$.

For the largest cliques on two vertices that are not maximal, we have $d_i(u) + d_i(v) = \omega(G_i) - 2$ because the demands are assumed to be even. Observe that the weights of these cliques are reduced by at least 10. This is true because it

is not possible to have $\tilde{d}_i(u) = \tilde{d}_i(v) = 4$ since $d_i(v) \leq \frac{\omega(G_i)}{3}$ and $d_i(u) \leq \frac{\omega(G_i)}{3}$ would imply $d_i(u) + d_i(v) \leq \frac{2\omega(G_i)}{3}$ contradicting the fact that $\omega(G_i) - 2 > \frac{2\omega(G_i)}{3}$ for $\omega(G_i) > 12$.

Clearly, the cliques on two vertices for which $d_i(u) + d_i(v) = \omega(G_i) - 4 = 8$ get at least $4+4=8$ colors.

Hence $\omega(G_{i+1}) \leq \omega(G_i) - 12$ as claimed.

Repeat application of the construction above to each G_i to obtain G_{i+1} , until $\omega(G_i) < 12$. From considerations above it follows that (1) the number of iterations is at most $\omega(G) \text{ div } 12$. (2) the total number of colors needed is at most $15(\omega(G) \text{ div } 12)$. Hence for $k = \omega(G) \text{ div } 12 = \lfloor \frac{\omega(G)}{12} \rfloor$ iterations we need $15k$ colors.

Finally, we have to color the last graph G_k with $\omega(G_k) < 12$ for which 15 colors suffice. \square

Theorem 3.2 *Let G be a weighted hexagonal graph. There is a proper multicoloring of G that uses at most $15\lfloor \frac{\omega(G)}{12} \rfloor + 18$ colors.*

Proof: Reduce the odd weights of vertices of G by one, and apply the previous theorem. There may be some vertices left that need one additional color. Use three additional colors (for example, the base colors) to fulfill the missing demand. \square

4 Conclusions and future work

The partition of a weighted hexagonal graph given in Subsection 3.1 was used to provide a 15 coloring of a class of graphs with clique number 12. This in turn enabled a $5/4$ approximation for general weighted hexagonal graphs.

The present author believes that the main idea, partitioning of hexagonal graph to triangle-free subgraphs, may be used to improve the bound $5/4$ further. There are $7/6$ approximation algorithms for triangle-free hexagonal graphs, however they can not be applied to the partition of weights as proposed in this paper. It is an interesting question whether it is possible to define another, more suitable partition.

More generally, an interesting open question is whether it is possible to partition the weights of a hexagonal graph G to three triangle-free subgraphs G_1 , G_2 , and G_3 such that $\omega(G_1) + \omega(G_2) + \omega(G_3) = \omega(G)$? Recall that the positive answer would imply that given any algorithm that gives a r -approximation for triangle-free hexagonal graphs we would have a r -approximation for general hexagonal graphs !

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